

Geometric Adaptive Control for Aerial Transportation of a Rigid Body

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Abstract—This paper is focused on tracking control for a rigid body payload, that is connected to an arbitrary number of quadrotor unmanned aerial vehicles via rigid links. A geometric adaptive controller is constructed such that the payload asymptotically follows a given desired trajectory for its position and attitude in the presence of uncertainties. The coupled dynamics between the rigid body payload, links, and quadrotors are explicitly incorporated into control system design and stability analysis. These are developed directly on the nonlinear configuration manifold in a coordinate-free fashion to avoid singularities and complexities that are associated with local parameterizations.

I. INTRODUCTION

By utilizing the high thrust-to-weight ratio, quadrotor unmanned aerial vehicles have been envisaged for aerial load transportation [1], [2], [3]. Most of the existing results for the control of quadrotors to transport a cable-suspended payload are based on the assumption that the dynamics of the payload is decoupled from the dynamics of quadrotors. For example, the effects of the payload are considered as arbitrary external force and torque exerted to quadrotors [2]. As such, these results may not be suitable for agile load transportation where the motion of cable and payload should be actively suppressed.

Recently, the full dynamic model for an arbitrary number of quadrotors transporting a payload are developed, and based on that, geometric tracking controllers are constructed in an intrinsic fashion. In particular, autonomous transportation of a point mass connected to quadrotors via rigid links is developed in [4]. It has been generalized into a more realistic dynamic model that considers the deformation of cables in [5], and also the attitude dynamics of a payload, that is considered as a rigid body instead of a point mass, is incorporated in [6]. However, these results are based on the assumption that the exact properties of the quadrotors and the payload are available, and that there are no external disturbances, thereby making it challenging to implement those results in actual hardware systems.

The objective of this paper is to construct a control system for an arbitrary number of quadrotors connected to a rigid body payload via rigid links with explicit consideration on uncertainties. A coordinate-free form of the equations of motion that have been developed in [6] is extended to include the effects of unknown, but fixed forces and moments acting on each of the quadrotors, the cables, and the payload. A geometric nonlinear adaptive control system

is designed such that both the position and the attitude of the payload asymptotically follow their desired trajectories, while maintaining a certain formation of quadrotors relative to the payload.

The unique property is that the coupled dynamics of the payload, the cables, and quadrotors are explicitly incorporated in control system design for agile load transportations where the motion of the payload relative to the quadrotors are excited nontrivially. Another distinct feature is that the equations of motion and the control systems are developed directly on the nonlinear configuration manifold intrinsically. Therefore, singularities of local parameterization are completely avoided.

As such, the proposed control system is particularly useful for rapid and safe payload transportation in complex terrain, where the position and attitude of the payload should be controlled concurrently. Most of the existing control systems of aerial load transportation suffer from limited agility as they are based on reactive assumptions that ignore the inherent complexities in the dynamics of aerial load transportation. The proposed control system explicitly integrates the comprehensive dynamic characteristics to achieve extreme maneuverability in aerial load transportation. To the author's best knowledge, nonlinear adaptive tracking controls of a cable-suspended rigid body with uncertainties have not been studied as mathematically rigorously as presented in this paper.

II. PROBLEM FORMULATION

Consider n quadrotor UAVs that are connected to a payload, that is modeled as a rigid body, via massless links (see Figure 1). Throughout this paper, the variables related to the payload is denoted by the subscript 0, and the variables for the i -th quadrotor are denoted by the subscript i , which is assumed to be an element of $\mathcal{I} = \{1, \dots, n\}$ if not specified. We choose an inertial reference frame $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ and body-fixed frames $\{\vec{b}_{j1}, \vec{b}_{j2}, \vec{b}_{j3}\}$ for $0 \leq j \leq n$ as follows. For the inertial frame, the third axis \vec{e}_3 points downward along the gravity and the other axes are chosen to form an orthonormal frame.

The location of the mass center of the payload is denoted by $x_0 \in \mathbb{R}^3$, and its attitude is given by $R_0 \in \text{SO}(3)$, where the special orthogonal group is defined by $\text{SO}(3) = \{R \in \mathbb{R}^{3 \times 3} \mid R^T R = I, \det[R] = 1\}$. Let $\rho_i \in \mathbb{R}^3$ be the point on the payload where the i -th link is attached, and it is represented with respect to the zeroth body-fixed frame. The other end of the link is attached to the mass center of the i -th quadrotor. The direction of the link from the mass center of the i -th quadrotor toward the payload is defined by the

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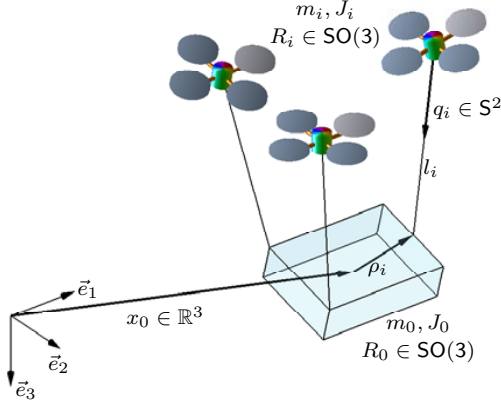


Fig. 1. Dynamics model: n quadrotors are connect to a rigid body m_0 via massless links l_i . The configuration manifold is $\mathbb{R}^3 \times \text{SO}(3) \times (\text{S}^2 \times \text{SO}(3))^n$.

unit-vector $q_i \in \text{S}^2$, where $\text{S}^2 = \{q \in \mathbb{R}^3 \mid \|q\| = 1\}$, and the length of the i -th link is denoted by $l_i \in \mathbb{R}$.

Let $x_i \in \mathbb{R}^3$ be the location of the mass center of the i -th quadrotor with respect to the inertial frame. As the link is assumed to be rigid, we have $x_i = x_0 + R_0 \rho_i - l_i q_i$. The attitude of the i -th quadrotor is defined by $R_i \in \text{SO}(3)$, which represents the linear transformation of the representation of a vector from the i -th body-fixed frame to the inertial frame.

In summary, the configuration of the presented system is described by the position x_0 and the attitude R_0 of the payload, the direction q_i of the links, and the attitudes R_i of the quadrotors. The corresponding configuration manifold of this system is $\text{Q} = \mathbb{R}^3 \times \text{SO}(3) \times (\text{S}^2 \times \text{SO}(3))^n$.

The mass and the inertia matrix of the payload are denoted by $m_0 \in \mathbb{R}$ and $J_0 \in \mathbb{R}^{3 \times 3}$, respectively. The dynamic model of each quadrotor is identical to [7]. The mass and the inertia matrix of the i -th quadrotor are denoted by $m_i \in \mathbb{R}$ and $J_i \in \mathbb{R}^{3 \times 3}$, respectively. The i -th quadrotor can generates a thrust $-f_i R_i e_3 \in \mathbb{R}^3$ with respect to the inertial frame, where $f_i \in \mathbb{R}$ is the total thrust magnitude and $e_3 = [0, 0, 1]^T \in \mathbb{R}^3$. It also generates a moment $M_i \in \mathbb{R}^3$ with respect to its body-fixed frame. The control input of this system corresponds to $\{f_i, M_i\}_{1 \leq i \leq n}$.

In this paper, the external disturbances are modeled as follows. The disturbance force and moment acting on the payload, namely $\Delta_{x_0}, \Delta_{R_0} \in \mathbb{R}^3$ are expressed as

$$\Delta_{x_0} = \Phi_{x_0}(t, q, \dot{q}) \theta_{x_0}, \quad \Delta_{R_0} = \Phi_{R_0}(t, q, \dot{q}) \theta_{R_0}, \quad (1)$$

where $\Phi_{x_0}, \Phi_{R_0} : \mathbb{R} \times \text{TQ} \rightarrow \mathbb{R}^{3 \times n_\theta}$ denote matrix-valued, known function of the time t and the tangent vector $(q, \dot{q}) \in \text{T}_q \text{Q}$ of the configuration manifold, i.e., $q = (x_0, R_0, q_1, \dots, q_n, R_1, \dots, R_n)$, and $\theta_{x_0}, \theta_{R_0} \in \mathbb{R}^{n_\theta \times 1}$ are fixed, unknown parameters for some n_θ . This type of uncertainties are popular in the literature of adaptive controls, and they may represent various modeling errors or disturbances, such as the uncertainties in the mass and the inertia matrix of the payload. Similarly, the disturbance force and moment acting on the i -th quadrotors are given by

$$\Delta_{x_i} = \Phi_{x_i}(t, q, \dot{q}) \theta_{x_i}, \quad \Delta_{R_i} = \Phi_{R_i}(t, q, \dot{q}) \theta_{R_i}, \quad (2)$$

where $\Phi_{x_i}, \Phi_{R_i} : \mathbb{R} \times \text{TQ} \rightarrow \mathbb{R}^{3 \times n_\theta}$ and $\theta_{x_i}, \theta_{R_i} \in \mathbb{R}^{n_\theta \times 1}$. Here, the disturbance forces are represented with respect to the inertial frame, and the disturbance moments are represented with respect to the corresponding body-fixed frame.

Throughout this paper, the 2-norm of a matrix A is denoted by $\|A\|$, and its maximum eigenvalue and minimum eigenvalues are denoted by $\lambda_M[A]$ and $\lambda_m[A]$, respectively. The standard dot product is denoted by $x \cdot y = x^T y$ for any $x, y \in \mathbb{R}^3$.

A. Equations of Motion

The kinematic equations for the payload, quadrotors, and links are given by

$$\dot{q}_i = \omega_i \times q_i = \hat{\omega}_i q_i, \quad (3)$$

$$\dot{R}_0 = R_0 \hat{\Omega}_0, \quad \dot{R}_i = R_i \hat{\Omega}_i, \quad (4)$$

where $\omega_i \in \mathbb{R}^3$ is the angular velocity of the i -th link, satisfying $q_i \cdot \omega_i = 0$, and Ω_0 and $\Omega_i \in \mathbb{R}^3$ are the angular velocities of the payload and the i -th quadrotor expressed with respect to its body-fixed frame, respectively. The *hat map* $\hat{\cdot} : \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ is defined by the condition that $\hat{x}y = x \times y$ for all $x, y \in \mathbb{R}^3$, and the inverse of the hat map is denoted by the *vee map* $\vee : \mathfrak{so}(3) \rightarrow \mathbb{R}^3$, where $\mathfrak{so}(3)$ denotes the set of 3×3 skew-symmetric matrices, i.e., $\mathfrak{so}(3) = \{S \in \mathbb{R}^{3 \times 3} \mid S^T = -S\}$, and it corresponds to the Lie algebra of $\text{SO}(3)$.

We derive equations of motion according to Lagrangian mechanics. The velocity of the i -th quadrotor is given by $\dot{x}_i = \dot{x}_0 + \dot{R}_0 \rho_i - l_i \dot{q}_i$. The kinetic energy of the system is composed of the translational kinetic energy and the rotational kinetic energy of the payload and quadrotors:

$$\begin{aligned} \mathcal{T} = & \frac{1}{2} m_0 \|\dot{x}_0\|^2 + \frac{1}{2} \Omega_0 \cdot J_0 \Omega_0 \\ & + \sum_{i=1}^n \left[\frac{1}{2} m_i \|\dot{x}_0 + \dot{R}_0 \rho_i - l_i \dot{q}_i\|^2 + \frac{1}{2} \Omega_i \cdot J_i \Omega_i \right]. \end{aligned} \quad (5)$$

The gravitational potential energy is given by

$$\mathcal{U} = -m_0 g e_3 \cdot x_0 - \sum_{i=1}^n m_i g e_3 \cdot (x_0 + R_0 \rho_i - l_i q_i), \quad (6)$$

where the unit-vector e_3 points downward along the gravitational acceleration as shown at Fig. 1. The resulting Lagrangian of the system is $\mathcal{L} = \mathcal{T} - \mathcal{U}$.

The corresponding Euler-Lagrange equations have been developed according to Hamilton's principle in [6]. Here, it is generalized to include the effects of disturbances via the Lagrange-d'Alembert principle. Let the action integral be $\mathfrak{S} = \int_{t_0}^{t_f} \mathcal{L} dt$. Next, let the total control thrust at the i -th quadrotor with respect to the inertial frame be denoted by $u_i = -f_i R_i e_3 \in \mathbb{R}^3$ and the total control moment at the i -th quadrotor is defined as $M_i \in \mathbb{R}^3$. There exist the disturbances $\Delta_{x_0}, \Delta_{R_0}$ for the payload, and the disturbances $\Delta_{x_i}, \Delta_{R_i}$ for the i -th quadrotor. The virtual work can be

written as

$$\begin{aligned} \delta\mathcal{W} = & \int_{t_0}^{t_f} \sum_{i=1}^n (u_i + \Delta_{x_i}) \cdot \{\delta x_0 + R_0 \hat{\eta}_0 \rho_i - l_i \xi_i \times q_i\} \\ & + \sum_{i=1}^n (M_i + \Delta_{R_i}) \cdot \eta_i + \Delta_{x_0} \cdot \delta x_0 + \Delta_{R_0} \cdot \eta_0 dt. \end{aligned}$$

The Lagrange-d'Alembert principle states that $\delta\mathfrak{G} = -\delta\mathcal{W}$ for any variation of trajectories with fixed end points. This yields the following equations of motion (see [8] for detailed derivations),

$$\begin{aligned} M_q(\ddot{x}_0 - ge_3) - \sum_{i=1}^n m_i q_i q_i^T R_0 \hat{\rho}_i \dot{\Omega}_0 &= \Delta_{x_0} \\ + \sum_{i=1}^n u_i^\parallel + \Delta_{x_i}^\parallel - m_i l_i \|\omega_i\|^2 q_i - m_i q_i q_i^T R_0 \hat{\Omega}_0^2 \rho_i, & \quad (7) \\ (J_0 - \sum_{i=1}^n m_i \hat{\rho}_i R_0^T q_i q_i^T R_0 \hat{\rho}_i) \dot{\Omega}_0 & \\ + \sum_{i=1}^n m_i \hat{\rho}_i R_0^T q_i q_i^T (\ddot{x}_0 - ge_3) + \hat{\Omega}_0 J_0 \Omega_0 &= \Delta_{R_0} \\ + \sum_{i=1}^n \hat{\rho}_i R_0^T (u_i^\parallel + \Delta_{x_i}^\parallel - m_i l_i \|\omega_i\|^2 q_i - m_i q_i q_i^T R_0 \hat{\Omega}_0^2 \rho_i), & \quad (8) \end{aligned}$$

$$\begin{aligned} \dot{\omega}_i &= \frac{1}{l_i} \hat{q}_i (\ddot{x}_0 - ge_3 - R_0 \hat{\rho}_i \dot{\Omega}_0 + R_0 \hat{\Omega}_0^2 \rho_i) \\ &\quad - \frac{1}{m_i l_i} \hat{q}_i (u_i^\perp + \Delta_{x_i}^\perp), \end{aligned} \quad (9)$$

$$J_i \dot{\Omega}_i + \Omega_i \times J_i \Omega_i = M_i + \Delta_{R_i}, \quad (10)$$

where $M_q = m_y I + \sum_{i=1}^n m_i q_i q_i^T \in \mathbb{R}^{3 \times 3}$, which is symmetric, positive-definite for any q_i .

Recall the vector $u_i \in \mathbb{R}^3$ represents the control force at the i -th quadrotor, i.e., $u_i = -f_i R_i e_3$. The vectors u_i^\parallel and $u_i^\perp \in \mathbb{R}^3$ denote the orthogonal projection of u_i along q_i , and the orthogonal projection of u_i to the plane normal to q_i , respectively, i.e.,

$$u_i^\parallel = q_i q_i^T u_i, \quad (11)$$

$$u_i^\perp = -\hat{q}_i^2 u_i = (I - q_i q_i^T) u_i. \quad (12)$$

Therefore, $u_i = u_i^\parallel + u_i^\perp$. Throughout this paper, the subscripts \parallel and \perp of a vector denote the component of the vector that is parallel to q_i and the other component of the vector that is perpendicular to q_i . Similarly, the disturbance force at the i -th quadrotor is decomposed as

$$\Delta_{x_i}^\parallel = q_i q_i^T \Phi_{x_i} \theta_{x_i} \triangleq \Phi_{x_i}^\parallel \theta_{x_i}, \quad (13)$$

$$\Delta_{x_i}^\perp = (I - q_i q_i^T) \Phi_{x_i} \theta_{x_i} \triangleq \Phi_{x_i}^\perp \theta_{x_i}. \quad (14)$$

B. Tracking Problem

Define a fixed matrix $\mathcal{P} \in \mathbb{R}^{6 \times 3n}$ as

$$\mathcal{P} = \begin{bmatrix} I_{3 \times 3} & \cdots & I_{3 \times 3} \\ \hat{\rho}_1 & \cdots & \hat{\rho}_n \end{bmatrix}. \quad (15)$$

Recall that ρ_i describe the point on the payload where the i -th link is attached. Assume the links are attached to the payload such that

$$\text{rank}[\mathcal{P}] \geq 6. \quad (16)$$

This is to guarantee that there exist enough degrees of freedom in control inputs for both the translational motion and the rotational maneuver of the payload. The assumption (16) requires that the number of quadrotor is at least three, i.e., $n \geq 3$.

It is also assumed that the bounds of the disturbance forces and moments are available, i.e., for known positive constant $B_\Phi, B_\theta \in \mathbb{R}$, we have

$$\begin{aligned} \max\{\|\Phi_{x_0}\|, \|\Phi_{R_0}\|, \|\Phi_{x_1}\|, \dots, \|\Phi_{x_n}\|, \\ \|\Phi_{R_0}\|, \dots, \|\Phi_{R_n}\|\} < B_\Phi, \end{aligned} \quad (17)$$

$$\begin{aligned} \max\{\|\theta_{x_0}\|, \|\theta_{R_0}\|, \|\theta_{x_1}\|, \dots, \|\theta_{x_n}\|, \\ \|\theta_{R_0}\|, \dots, \|\theta_{R_n}\|\} < B_\theta. \end{aligned} \quad (18)$$

Suppose that the desired trajectories for the position and the attitude of the payload are given as smooth functions of time, namely $x_{0_d}(t) \in \mathbb{R}^3$ and $R_{0_d}(t) \in \text{SO}(3)$. From the attitude kinematics equation, we have

$$\dot{R}_{0_d}(t) = R_{0_d}(t) \hat{\Omega}_{0_d}(t),$$

where $\Omega_{0_d}(t) \in \mathbb{R}^3$ corresponds to the desired angular velocity of the payload. It is assumed that the velocity and the acceleration of the desired trajectories are bounded by known constants.

We wish to design a control input of each quadrotor $\{f_i, M_i\}_{1 \leq i \leq n}$ such that the tracking errors asymptotically converge to zero along the solution of the controlled dynamics.

III. CONTROL SYSTEM DESIGN FOR SIMPLIFIED DYNAMIC MODEL

In this section, we consider a simplified dynamic model where the attitude dynamics of each quadrotor is ignored, and we design a control input by assuming that the thrust at each quadrotor, namely u_i can be arbitrarily chosen. It corresponds to the case where every quadrotor is replaced by a fully actuated aerial vehicle that can generate a thrust along any direction arbitrarily. The effects of the attitude dynamics of quadrotors will be incorporated in the next section.

In the simplified dynamic model given by (7)-(9), the dynamics of the payload are affected by the parallel components u_i^\parallel of the thrusts, and the dynamics of the links are directly affected by the normal components u_i^\perp of the thrusts. This structure motivates the following control system design procedure: first, the parallel components u_i^\parallel are chosen such that the payload follows the desired position and attitude trajectory while yielding the desired direction of each link, namely $q_{i_d} \in \mathbb{S}^2$; next, the normal components u_i^\perp are designed such that the actual direction of the links q_i follows the desired direction q_{i_d} .

A. Design of Parallel Components

Let $a_i \in \mathbb{R}^3$ be the acceleration of the point on the payload where the i -th link is attached, that is measured relative to the gravitational acceleration:

$$a_i = \ddot{x}_0 - ge_3 + R_0 \hat{\Omega}_0^2 \rho_i - R_0 \hat{\rho}_i \dot{\Omega}_0. \quad (19)$$

The parallel component of the control input is chosen as

$$u_i^\parallel = \mu_i + m_i l_i \|\omega_i\|^2 q_i + m_i q_i q_i^T a_i, \quad (20)$$

where $\mu_i \in \mathbb{R}^3$ is a virtual control input that is designed later, with a constraint that μ_i is parallel to q_i . Note that the expression of u_i^\parallel is guaranteed to be parallel to q_i due to the projection operator $q_i q_i^T$ at the last term of the right-hand side of the above expression.

The motivation for the proposed parallel components becomes clear if (20) is substituted into (7)-(8) and rearranged to obtain

$$m_0(\ddot{x}_0 - ge_3) = \Delta_{x_0} + \sum_{i=1}^n (\mu_i + \Delta_{x_i}^\parallel), \quad (21)$$

$$J_0 \dot{\Omega}_0 + \hat{\Omega}_0 J_0 \Omega_0 = \Delta_{R_0} + \sum_{i=1}^n \hat{\rho}_i R_0^T (\mu_i + \Delta_{x_i}^\parallel). \quad (22)$$

Therefore, considering a free-body diagram of the payload, the virtual control input μ_i corresponds to the force exerted to the payload by the i -link, or the tension of the i -th link in the absence of disturbances.

Next, we determine the virtual control input μ_i . As in [9], define position, attitude, and angular velocity tracking error vectors $e_{x_0}, e_{R_0}, e_{\Omega_0} \in \mathbb{R}^3$ for the payload as

$$\begin{aligned} e_{x_0} &= x_0 - x_{0d}, \\ e_{R_0} &= \frac{1}{2}(R_{0d}^T R_0 - R_0^T R_{0d})^\vee, \\ e_{\Omega_0} &= \Omega_0 - R_{0d}^T R_{0d} \Omega_{0d}. \end{aligned}$$

The desired resultant control force $F_d \in \mathbb{R}^3$ and moment $M_d \in \mathbb{R}^3$ acting on the payload are given as

$$\begin{aligned} F_d &= m_0(-k_{x_0} e_{x_0} - k_{\dot{x}_0} \dot{e}_{x_0} + \ddot{x}_{0d} - ge_3) \\ &\quad - \Phi_{x_0} \bar{\theta}_{x_0} - \sum_{i=1}^n \Phi_{x_i}^\parallel \bar{\theta}_{x_i}, \end{aligned} \quad (23)$$

$$\begin{aligned} M_d &= -k_{R_0} e_{R_0} - k_{\Omega_0} e_{\Omega_0} + (R_0^T R_{0d} \Omega_{0d})^\wedge J_0 R_0^T R_{0d} \Omega_{0d} \\ &\quad + J_0 R_0^T R_{0d} \dot{\Omega}_{0d} - \Phi_{R_0} \bar{\theta}_{R_0} - \sum_{i=1}^n \hat{\rho}_i R_0 \Phi_{x_i}^\parallel \bar{\theta}_{x_i}, \end{aligned} \quad (24)$$

for positive constants $k_{x_0}, k_{\dot{x}_0}, k_{R_0}, k_{\Omega_0} \in \mathbb{R}$. Here, the estimates of the unknown parameters $\theta_{x_0}, \theta_{x_i}, \theta_{R_0}$ are denoted by $\bar{\theta}_{x_0}, \bar{\theta}_{x_i}, \bar{\theta}_{R_0} \in \mathbb{R}^{n_\theta}$. Adaptive control laws to update the estimates of disturbances are introduced later at Section III-C.

These are the ideal resultant force and moment to achieve the control objectives. One may try to choose the virtual control input μ_i by making the expressions in the right-hand sides of (21) and (22), namely $\sum_i \mu_i$ and $\sum_i \hat{\rho}_i R_0^T \mu_i$, become identical to F_d and M_d , respectively. But, this is not

valid in general, as each μ_i is constrained to be parallel to q_i . Instead, we choose the desired value of μ_i , without any constraint, such that

$$\sum_{i=1}^n \mu_{i_d} = F_d, \quad \sum_{i=1}^n \hat{\rho}_i R_0^T \mu_{i_d} = M_d, \quad (25)$$

or equivalently, using the matrix \mathcal{P} defined at (15),

$$\mathcal{P} \begin{bmatrix} R_0^T \mu_{1_d} \\ \vdots \\ R_0^T \mu_{n_d} \end{bmatrix} = \begin{bmatrix} R_0^T F_d \\ M_d \end{bmatrix}.$$

From the assumption stated at (16), there exists at least one solution to the above matrix equation for any F_d, M_d . Here, we find the minimum-norm solution given by

$$\begin{bmatrix} \mu_{1_d} \\ \vdots \\ \mu_{n_d} \end{bmatrix} = \text{diag}[R_0, \dots, R_0] \mathcal{P}^T (\mathcal{P} \mathcal{P}^T)^{-1} \begin{bmatrix} R_0^T F_d \\ M_d \end{bmatrix}. \quad (26)$$

The virtual control input μ_i is selected as the projection of its desired value μ_{i_d} along q_i ,

$$\mu_i = (\mu_{i_d} \cdot q_i) q_i = q_i q_i^T \mu_{i_d}, \quad (27)$$

and the desired direction of each link, namely $q_{i_d} \in \mathbb{S}^2$ is defined as

$$q_{i_d} = -\frac{\mu_{i_d}}{\|\mu_{i_d}\|}. \quad (28)$$

It is straightforward to verify that when $q_i = q_{i_d}$, the resultant force and moment acting on the payload become identical to their desired values.

B. Design of Normal Components

Substituting (19) into (9) and using (14), the equation of motion for the i -link is given by

$$\dot{\omega}_i = \frac{1}{l_i} \hat{q}_i a_i - \frac{1}{m_i l_i} \hat{q}_i (u_i^\perp + \Delta_{x_i}^\perp). \quad (29)$$

Here, the normal component of the control input u_i^\perp is chosen such that $q_i \rightarrow q_{i_d}$ as $t \rightarrow \infty$. Control systems for the unit-vectors on the two-sphere have been studied in [10], [11]. In this paper, we adopt the control system developed in terms of the angular velocity in [11], and we augment it with an adaptive control term to handle the disturbance $\Delta_{x_i}^\perp$.

For the given desired direction of each link, its desired angular velocity is obtained from the kinematics equation as

$$\omega_{i_d} = q_{i_d} \times \dot{q}_{i_d}.$$

Define the direction and the angular velocity tracking error vectors for the i -th link, namely $e_{q_i}, e_{\omega_i} \in \mathbb{R}^3$ as

$$\begin{aligned} e_{q_i} &= q_{i_d} \times q_i, \\ e_{\omega_i} &= \omega_i + \hat{q}_i^2 \omega_{i_d}. \end{aligned}$$

For positive constants $k_q, k_\omega \in \mathbb{R}$, the normal component of the control input is chosen as

$$u_i^\perp = m_i l_i \hat{q}_i \{-k_q e_{q_i} - k_\omega e_{\omega_i} - (q_i \cdot \omega_{i_d}) \dot{q}_i - \hat{q}_i^2 \dot{\omega}_d\}$$

$$-m_i \hat{q}_i^2 a_i - \Phi_{x_i}^\perp \bar{\theta}_{x_i}. \quad (30)$$

Note that the expression of u_i^\perp is perpendicular to q_i by definition. Substituting (30) into (29), and rearranging by the facts that the matrix $-\hat{q}_i^2$ corresponds to the orthogonal projection to the plane normal to q_i and $\hat{q}_i^3 = -\hat{q}_i$, we obtain

$$\begin{aligned} \dot{\omega}_i &= -k_q e_{q_i} - k_\omega e_{\omega_i} - (q_i \cdot \omega_{i_d}) \dot{q}_i - \hat{q}_i^2 \dot{\omega}_d \\ &\quad - \frac{1}{m_i l_i} \hat{q}_i \Phi_{x_i}^\perp \tilde{\theta}_{x_i}, \end{aligned} \quad (31)$$

where the estimation error is defined as $\tilde{\theta}_{x_i}^\perp = \theta_{x_i} - \bar{\theta}_{x_i} \in \mathbb{R}^{n_\theta}$.

In short, the control force for the simplified dynamic model is given by

$$u_i = u_i^\parallel + u_i^\perp. \quad (32)$$

C. Design of Adaptive Law

Next, we design the adaptive laws to construct the estimates of unknown parameters. The following projection operator is introduced such that the estimated parameters stay in the bound of the true parameters given by (18).

$$\text{Pr}(\bar{\theta}, y) = \begin{cases} y & \text{if } \|\bar{\theta}\| < B_\theta \\ \text{or } \|\bar{\theta}\| = B_\theta \text{ and } \bar{\theta}^T y \leq 0, \\ (I_{n_\theta \times n_\theta} - \frac{1}{\|\bar{\theta}\|^2} \bar{\theta} \bar{\theta}^T) y & \text{otherwise.} \end{cases} \quad (33)$$

Using this, the adaptive laws are defined as

$$\dot{\bar{\theta}}_{x_0} = \text{Pr}(\bar{\theta}_{x_0}, y_{x_0}), \quad (34)$$

$$\dot{\bar{\theta}}_{R_0} = \text{Pr}(\bar{\theta}_{R_0}, y_{R_0}), \quad (35)$$

$$\dot{\bar{\theta}}_{x_i} = \text{Pr}(\bar{\theta}_{x_i}, y_{x_i}), \quad (36)$$

where $y_{x_0}, y_{R_0}, y_{x_i} \in \mathbb{R}^{n_\theta}$ are defined as

$$y_{x_0} = \frac{h_{x_0}}{m_0} \Phi_{x_0}^T (\dot{e}_{x_0} + c_x e_{x_0}), \quad (37)$$

$$y_{R_0} = h_{R_0} \Phi_{R_0}^T (e_{\Omega_0} + c_R e_{R_0}), \quad (38)$$

$$\begin{aligned} y_{x_i} &= h_{x_i} \Phi_{x_0}^T [q_i q_i^T \{ \frac{1}{m_0} (\dot{e}_{x_0} + c_x e_{x_0}) \\ &\quad - R_0 \hat{\rho}_i (e_{\Omega_0} + c_R e_{R_0}) \} + \frac{1}{m_i l_i} \hat{q}_i (e_{\omega_i} + c_q e_{q_i})], \end{aligned} \quad (39)$$

for positive constants $c_x, c_R, c_q \in \mathbb{R}$ and adaptive gains $h_{x_0}, h_{R_0}, h_{x_i} \in \mathbb{R}$.

The first case of the projection map is the identity map, and the second case corresponds to the case that the estimated parameters are at the boundary of the region defined by (18) and the unprojected direction y for the change of the estimates points outward. For such cases, y is projected onto the plane tangent to the boundary such that the estimated parameters remain on the region [12].

The resulting stability properties are summarized as follows.

Proposition 1: Consider the simplified dynamic model defined by (7)-(9). For given tracking commands x_{0_d}, R_{0_d} , a control input is designed as (32)-(36). Then, there exist the

values of controller gains and controller parameters such that the following properties are satisfied.

- (i) The zero equilibrium of tracking errors $(e_{x_0}, \dot{e}_{x_0}, e_{R_0}, e_{\Omega_0}, e_{q_i}, e_{\omega_i})$ and the estimation errors $(\theta_{x_0}, \theta_{R_0}, \theta_{x_i})$ is stable in the sense of Lyapunov.
- (ii) The tracking errors asymptotically coverage to zero.
- (iii) The magnitude of the estimated parameters is less than B_θ always, provided that the magnitude of their initial estimates is less than B_θ .

Proof: Due to the page limit, the proof is relegated to [8]. ■

IV. CONTROL SYSTEM DESIGN FOR FULL DYNAMIC MODEL

The control system designed at the previous section is based on a simplifying assumption that each quadrotor can generate a thrust along any arbitrary direction instantaneously. However, the dynamics of quadrotor is under-actuated since the direction of the total thrust is always parallel to its third body-fixed axis, while the magnitude of the total thrust can be arbitrarily changed. This can be directly observed from the expression of the total thrust, $u_i = -f_i R_i e_3$, where f_i is the total thrust magnitude, and $R_i e_3$ corresponds to the direction of the third body-fixed axis. Whereas, the rotational attitude dynamics is fully actuated by the control moment M_i .

Based on these observations, the attitude of each quadrotor is controlled such that the third body-fixed axis becomes parallel to the direction of the ideal control force u_i designed in the previous section within a finite time. More explicitly, the desired attitude of each quadrotor is constructed as follows. The desired direction of the third body-fixed axis of the i -th quadrotor, namely $b_{3_i} \in \mathbb{S}^2$ is given by

$$b_{3_i} = -\frac{u_i}{\|u_i\|}. \quad (40)$$

This provides two-dimensional constraint on the three-dimensional desired attitude of each quadrotor, and there remains one degree of freedom. To resolve it, the desired direction of the first body-fixed axis $b_{1_i}(t) \in \mathbb{S}^2$ is introduced as a smooth function of time [7]. This corresponds to controlling the additional one dimensional yawing angle of each quadrotor. From these, the desired attitude of the i -th quadrotor is given by

$$R_{i_c} = \left[-\frac{(\hat{b}_{3_i})^2 b_{1_i}}{\|(\hat{b}_{3_i})^2 b_{1_i}\|}, \frac{\hat{b}_{3_i} b_{1_i}}{\|\hat{b}_{3_i} b_{1_i}\|}, b_{3_i} \right],$$

which is guaranteed to be an element of $\text{SO}(3)$. The desired angular velocity is obtained from the attitude kinematics equation, $\Omega_{i_c} = (R_{i_c}^T \dot{R}_{i_c})^\vee \in \mathbb{R}^3$.

In the prior work described in [6], the attitude of each quadrotor is controlled such that the equilibrium $R_i = R_{i_c}$ becomes exponentially stable, and the stability of the combined full dynamic model is achieved via singular perturbation theory [13]. However, we can not follow such approach in this paper, as the presented adaptive control system guarantees only the asymptotical convergence of the tracking error variables due to the disturbances, thereby making it

challenging to apply the singular perturbation theory. Here, we design the attitude controller of each quadrotor such that R_i becomes equal to R_{i_c} within a finite time via finite-time stability theory [14], [15], [16].

Define the tracking error vectors $e_{R_i}, e_{\Omega_i} \in \mathbb{R}^3$ for the attitude and the angular velocity of the i -th quadrotor as

$$e_{R_i} = \frac{1}{2}(R_{i_c}^T R_i - R_i^T R_{i_c})^\vee, \quad e_{\Omega_i} = \Omega_i - R_i^T R_{i_c} \Omega_{i_c}.$$

The time-derivative of e_{R_i} can be written as [7]

$$\dot{e}_{R_i} = \frac{1}{2}(\text{tr}[R_i^T R_{i_c}] I - R_i^T R_{i_c}) e_{\Omega_i} \triangleq E(R_i, R_{i_c}) e_{\Omega_i}. \quad (41)$$

For $0 < r < 1$, define $S : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ as

$$S(r, y) = [|y_1|^r \text{sgn}(y_1), |y_2|^r \text{sgn}(y_2), |y_3|^r \text{sgn}(y_3)]^T,$$

where $y = [y_1, y_2, y_3]^T \in \mathbb{R}^3$, and $\text{sgn}(\cdot)$ denotes the sign function. For positive constants k_R, l_R , the terminal sliding surface $s_i \in \mathbb{R}^3$ is designed as

$$s_i = e_{\Omega_i} + k_R e_{R_i} + l_R S(r, e_{R_i}). \quad (42)$$

We can show that when confined to the surface of $s_i \equiv 0$, the tracking errors become zero in a finite time. To reach the sliding surface, for positive constants k_s, l_s , the control moment is designed as

$$\begin{aligned} M_i = & -k_s s_i - l_s S(r, s_i) + \Omega_i \times J_i \Omega_i \\ & - (k_R J_i + l_s r J_i \text{diag}_j[|e_{R_{i_j}}|^{r-1}]) E(R_i, R_{i_c}) e_{\Omega_i} \\ & - J_i (\hat{\Omega}_i R_i^T R_{i_c} \Omega_{i_c} - R_i^T R_{i_c} \dot{\Omega}_{i_c}). \end{aligned} \quad (43)$$

The thrust magnitude is chosen as the length of u_i , projected on to $-R_i e_3$,

$$f_i = -u_i \cdot R_i e_3, \quad (44)$$

which yields that the thrust of each quadrotor becomes equal to its desired value u_i when $R_i = R_{i_c}$.

Stability of the corresponding controlled systems for the full dynamic model can be shown by using the fact that the full dynamic model becomes exactly same as the simplified dynamic model within a finite time.

Proposition 2: Consider the full dynamic model defined by (7)-(10). For given tracking commands x_{0_d}, R_{0_d} and the desired direction of the first body-fixed axis b_{1_i} , control inputs for quadrotors are designed as (43) and (44). Then, there exists controller parameters such that the tracking error variables $(e_{x_0}, \dot{e}_{x_0}, e_{R_0}, e_{\Omega_0}, e_{q_i}, e_{\omega_i})$ asymptotically converge to zero, and the estimation errors are uniformly bounded.

Proof: See Appendix A. ■

This implies that the payload asymptotically follows any arbitrary desired trajectory both in translations and rotations in the presence of uncertainties. In contrast to the existing results in aerial transportation of a cable suspended load, it does not rely on any simplifying assumption that ignores the coupling between payload, cable, and quadrotors. Also, the presented global formulation on the nonlinear

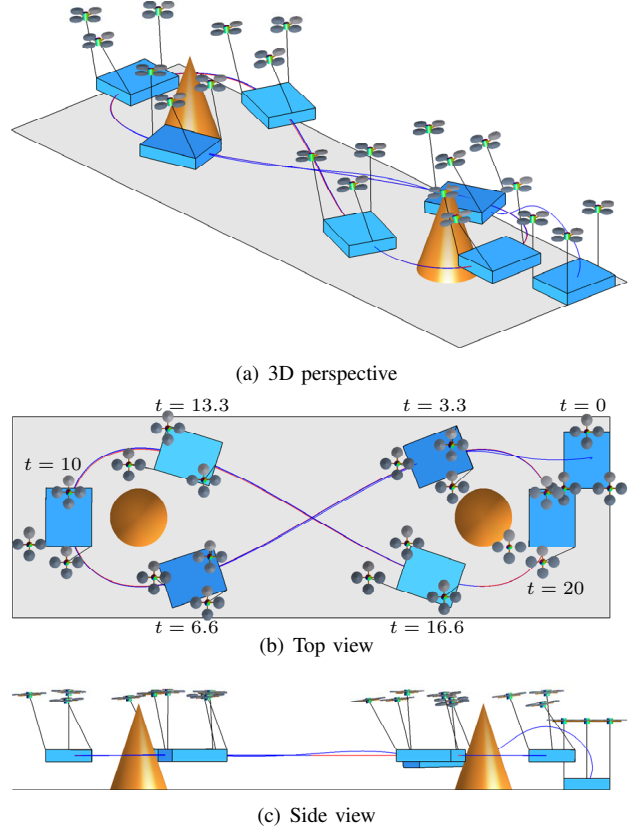


Fig. 2. Snapshots of controlled maneuver (red:desired trajectory, blue:actual trajectory). A short animation illustrating this maneuver is available at <http://youtu.be/nOWErfdzZLU>.

configuration manifold avoids singularities and complexities that are inherently associated with local coordinates. As such, the presented control system is particularly useful for agile load transportation involving combined translational and rotational maneuvers of the payload in the presence of uncertainties.

V. NUMERICAL EXAMPLE

We consider a numerical example where three quadrotors ($n = 3$) transport a rectangular box along a figure-eight curve. More explicitly, the mass of the payload is $m_0 = 1.5$ kg, and its length, width, and height are 1.0 m, 0.8 m, and 0.2 m, respectively. Mass properties of three quadrotors are identical, and they are given by

$$m_i = 0.755 \text{ kg}, \quad J_i = \text{diag}[0.0820, 0.0845, 0.1377] \text{ kgm}^2.$$

The length of cable is $l_i = 1$ m, and they are attached to the following points of the payload.

$$\begin{aligned} \rho_1 &= [0.5, 0, -0.1]^T, \\ \rho_1 &= [-0.5, 0.4, -0.1]^T, \quad \rho_3 = [-0.5, -0.4, -0.1]^T \text{ m.} \end{aligned}$$

In other words, the first link is attached to the center of the top, front edge, and the remaining two links are attached to the vertices of the top, rear edge (see Figure 1).

The desired trajectory of the payload is chosen as

$$x_{0_d}(t) = [1.2 \sin(0.2\pi t), 4.2 \cos(0.1\pi t), -0.5]^T \text{ m.}$$

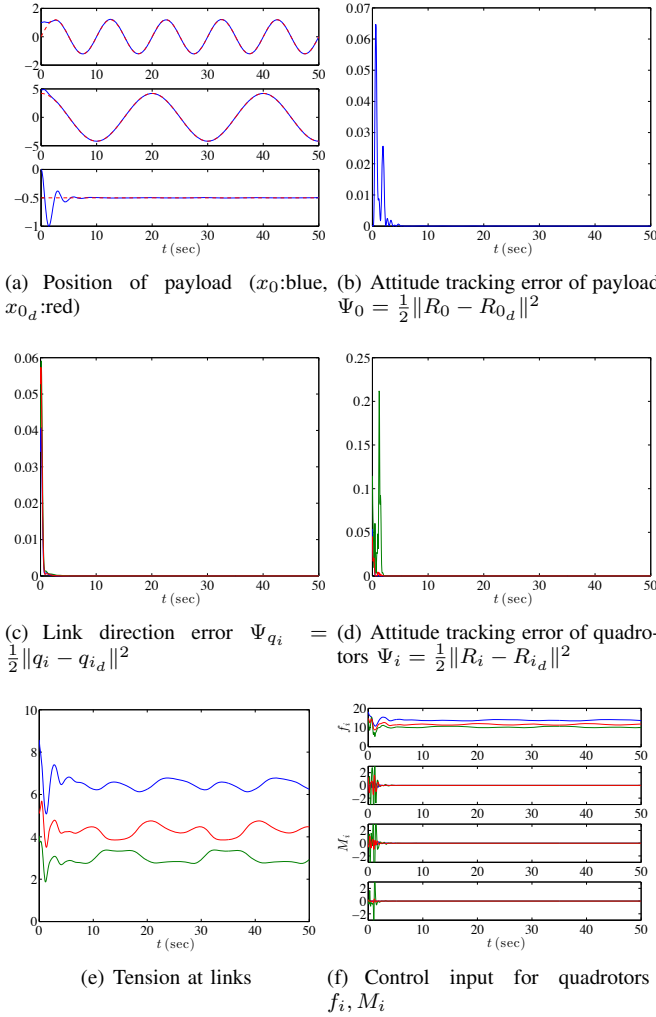


Fig. 3. Simulation results for tracking errors and control inputs. (for figures (c)-(f): $i = 1$:blue, $i = 2$:green, $i = 3$:red)

The desired attitude of the payload is chosen such that its first axis is tangent to the desired path, and the third axis is parallel to the direction of gravity, it is given by

$$R_{0_d}(t) = \begin{bmatrix} \dot{x}_{0_d} & \frac{\dot{e}_3 \dot{x}_{0_d}}{\|\dot{e}_3 \dot{x}_{0_d}\|} & e_3 \end{bmatrix}.$$

Initial conditions are chosen as

$$x_0(0) = [1, 4.8, 0]^T \text{ m}, \quad v_0(0) = 0_{3 \times 1} \text{ m/s}, \\ q_i(0) = e_3, \quad \omega_i(0) = 0_{3 \times 1}, \quad R_i(0) = I_{3 \times 3}, \quad \Omega_i(0) = 0_{3 \times 1}.$$

The uncertainties are specified as

$$\Delta_{x_0} = [1, 3, -2.5]^T, \quad \Delta_{R_0} = [-0.5, 0.1, -1.5]^T, \\ \Delta_{x_i} = [0.5, -0.2, 0.3]^T, \quad \Delta_{R_i} = [0.2, 0.3, -0.7]^T.$$

The corresponding simulation results are presented at Figures 2 and 3. Figure 2 illustrates the desired trajectory that is shaped like a figure-eight curve around two obstacles represented by cones, and the actual maneuver of the payload and quadrotors. Figure 3 shows tracking errors for the position and the attitude of the payload, tracking errors for the link directions and the attitude of quadrotors, as well as

tension and control inputs. These illustrate excellent tracking performances of the proposed control system.

APPENDIX

a) Error Dynamics: From (21) and (27), the dynamics of the position tracking error is given by

$$m_0 \ddot{e}_{x_0} = m_0 (ge_3 - \ddot{x}_{0_d}) + \Delta_{x_0} + \sum_{i=1}^n (q_i q_i^T \mu_{i_d} + \Delta_{x_i}^{\parallel}).$$

From (23) and (25), this can be rearranged as

$$\begin{aligned} \ddot{e}_{x_0} &= ge_3 - \ddot{x}_{0_d} + \frac{1}{m_0} F_d + Y_x + \frac{1}{m_0} (\Delta_{x_0} + \sum_{i=1}^n \Delta_{x_i}^{\parallel}), \\ &= -k_{x_0} e_{x_0} - k_{\dot{x}_0} \dot{e}_{x_0} + \frac{1}{m_0} (\Phi_{x_0} \tilde{\theta}_{x_0} + \sum_{i=1}^n \Phi_{x_i}^{\parallel} \tilde{\theta}_{x_i}) + Y_x, \end{aligned} \quad (45)$$

where (13) is used and the estimation error is denoted by $\tilde{\theta}_{x_i} = \theta_{x_i} - \hat{\theta}_{x_i}$. At the above equation, the last term $Y_x \in \mathbb{R}^3$ represents the error caused by the difference between q_i and q_{i_d} , and it is given by

$$Y_x = \frac{1}{m_0} \sum_{i=1}^n (q_i q_i^T - I) \mu_{i_d}.$$

We have $\mu_{i_d} = q_{i_d} q_{i_d}^T \mu_{i_d}$ from (28). Using this, the error term can be written in terms of e_{q_i} as

$$\begin{aligned} Y_x &= \frac{1}{m_0} \sum_{i=1}^n (q_{i_d}^T \mu_{i_d}) \{ (q_i^T q_{i_d}) q_i - q_{i_d} \} \\ &= -\frac{1}{m_0} \sum_{i=1}^n (q_{i_d}^T \mu_{i_d}) \hat{q}_i e_{q_i}. \end{aligned}$$

Using (26), an upper bound of Y_x can be obtained as

$$\|Y_x\| \leq \frac{1}{m_0} \sum_{i=1}^n \|\mu_{i_d}\| \|e_{q_i}\| \leq \sum_{i=1}^n \gamma (\|F_d\| + \|M_d\|) \|e_{q_i}\|,$$

where $\gamma = \frac{1}{m_0 \sqrt{\lambda_m(\mathcal{P}\mathcal{P}^T)}}$. From (23) and (24), this can be further bounded by

$$\begin{aligned} \|Y_x\| &\leq \sum_{i=1}^n \{ \beta (k_{x_0} \|e_{x_0}\| + k_{\dot{x}_0} \|\dot{e}_{x_0}\|) \\ &\quad + \gamma (k_{R_0} \|e_{R_0}\| + k_{\Omega_0} \|e_{\Omega_0}\|) + B \} \|e_{q_i}\|, \end{aligned} \quad (46)$$

where $\beta = m_0 \gamma$, and the constant B is determined by the given desired trajectories of the payload, the assumption (17) on the bounds of Φ terms, and the adaptive law defined later that guarantee the boundedness of the estimated parameters $\hat{\theta}$. Throughout the remaining parts of the proof, any bound that can be obtained from x_{0_d} , R_{0_d} , (17), or the adaptive law is denoted by B for simplicity. In short, the position tracking error dynamics of the payload can be written as (45), where the error term is bounded by (46).

Similarly, we find the attitude tracking error dynamics for the payload as follows. Using (22), (24), and (27), the time-derivative of $J_0 e_{\Omega_0}$ can be written as

$$J_0 \dot{e}_{\Omega_0} = (J_0 e_{\Omega_0} + d)^\wedge e_{\Omega_0} - k_{R_0} e_{R_0} - k_{\Omega_0} e_{\Omega_0}$$

$$+ \Phi_{R_0} \tilde{\theta}_{R_0} + \sum_{i=1}^n \hat{\rho}_i R_0^T \Phi_{R_0}^{\parallel} \tilde{\theta}_{x_i} + Y_R, \quad (47)$$

where $d = (2J_0 - \text{tr}[J_0] I) R_0^T R_{0_d} \Omega_{0_d} \in \mathbb{R}^3$ [9] that is bounded, and $\tilde{\Delta}_{R_0} \in \mathbb{R}^3$ denotes the estimation error given by $\tilde{\Delta}_{R_0} = \Delta_{R_0} - \hat{\Delta}_{R_0}$. The error term in the attitude dynamics of the payload, namely $Y_R \in \mathbb{R}^3$ is given by

$$Y_R = \sum_{i=1}^n \hat{\rho}_i R_0^T (q_i q_i^T - I) \mu_{i_d} = - \sum_{i=1}^n \hat{\rho}_i R_0^T (q_i^T \mu_{i_d}) \hat{q}_i e_{q_i}.$$

Similar with (46), an upper bound of Y_R can be obtained as

$$\|Y_R\| \leq \sum_{i=1}^n \{ \delta_i (k_{x_0} \|e_{x_0}\| + k_{\dot{x}_0} \|\dot{e}_{x_0}\|) + \sigma_i (k_{R_0} \|e_{R_0}\| + k_{\Omega_0} \|e_{\Omega_0}\|) + B \} \|e_{q_i}\|, \quad (48)$$

where $\delta_i = m_0 \frac{\|\hat{\rho}_i\|}{\sqrt{\lambda_m[\mathcal{P}\mathcal{P}^T]}}$, $\sigma_i = \frac{\delta_i}{m_0} \in \mathbb{R}$.

Next, from (31), the time-derivative of the angular velocity error, projected on to the plane normal to q_i is given as

$$\begin{aligned} -\dot{\hat{q}}_i^2 \dot{e}_{\omega_i} &= \dot{\omega} + (q \cdot \omega_d) \dot{q} + \hat{q}^2 \dot{\omega}_d \\ &= -k_q e_{q_i} - k_{\omega} e_{\omega_i} - \frac{1}{m_i l_i} \hat{q}_i \Phi_{x_i}^{\perp} \tilde{\theta}_{x_i}. \end{aligned} \quad (49)$$

In summary, the error dynamics of the simplified dynamic model are given by (45), (47) and (49).

b) Stability Proof: Define an attitude configuration error function Ψ_{R_0} for the payload as

$$\Psi_{R_0} = \frac{1}{2} \text{tr}[I - R_{0_d}^T R_0],$$

which is positive-definite about $R_0 = R_{0_d}$, and $\dot{\Psi}_{R_0} = e_{R_0} \cdot e_{\Omega_0}$ [7], [9]. We also introduce a configuration error function Ψ_{q_i} for each link that is positive-definite about $q_i = q_{i_d}$ as

$$\Psi_{q_i} = 1 - q_i \cdot q_{i_d}.$$

For positive constants $e_{x_{\max}}, \psi_{R_0}, \psi_{q_i}$, consider the following open domain containing the zero equilibrium of tracking error variables:

$$\begin{aligned} D = \{ & (e_{x_0}, \dot{e}_{x_0}, e_{R_0}, e_{\Omega_0}, e_{q_i}, e_{\omega_i}, \tilde{\Delta}_{x_0}, \tilde{\Delta}_{R_0}, \tilde{\Delta}_{x_i}) \\ & \in (\mathbb{R}^3)^4 \times (\mathbb{R}^3 \times \mathbb{R}^3)^n \times (\mathbb{R}^3)^2 \times \mathbb{R}^{3n} \mid \\ & \|e_{x_0}\| < e_{x_{\max}}, \Psi_{R_0} < \psi_{R_0} < 1, \Psi_{q_i} < \psi_{q_i} < 1 \}. \end{aligned} \quad (50)$$

In this domain, we have $\|e_{R_0}\| = \sqrt{\Psi_{R_0}(2 - \Psi_{R_0})} \leq \sqrt{\psi_{R_0}(2 - \psi_{R_0})} \triangleq \alpha_0 < 1$, and $\|e_{q_i}\| = \sqrt{\Psi_{q_i}(2 - \Psi_{q_i})} \leq \sqrt{\psi_{q_i}(2 - \psi_{q_i})} \triangleq \alpha_i < 1$. It is assumed that ψ_{q_i} is sufficiently small such that $n\alpha_i\beta < 1$.

We can show that the configuration error functions are quadratic with respect to the error vectors in the sense that

$$\begin{aligned} \frac{1}{2} \|e_{R_0}\|^2 &\leq \Psi_{R_0} \leq \frac{1}{2 - \psi_{R_0}} \|e_{R_0}\|^2, \\ \frac{1}{2} \|e_{q_i}\|^2 &\leq \Psi_{q_i} \leq \frac{1}{2 - \psi_{q_i}} \|e_{q_i}\|^2, \end{aligned}$$

where the upper bounds are satisfied only in the domain D .

Define

$$\begin{aligned} \mathcal{V}_0 &= \frac{1}{2} \|\dot{e}_{x_0}\|^2 + \frac{1}{2} k_{x_0} \|e_{x_0}\|^2 + c_x e_{x_0} \cdot \dot{e}_{x_0} \\ &\quad + \frac{1}{2} e_{\Omega_0} \cdot J_0 \Omega_0 + k_{R_0} \Psi_{R_0} + c_R e_{R_0} \cdot J_0 e_{\Omega_0} \\ &\quad + \sum_{i=1}^n \frac{1}{2} \|e_{\omega_i}\|^2 + k_q \Psi_{q_i} + c_q e_{q_i} \cdot e_{\omega_i}, \end{aligned}$$

where c_x, c_R, c_q are positive constants. This is composed of tracking error variables only, and we define another function for the estimation errors of the adaptive laws as

$$\mathcal{V}_a = \frac{1}{2h_{x_0}} \|\tilde{\theta}_{x_0}\|^2 + \frac{1}{2h_{R_0}} \|\tilde{\theta}_{R_0}\|^2 + \sum_{i=1}^n \frac{1}{2h_{x_i}} \|\tilde{\theta}_{x_i}\|^2.$$

The Lyapunov function for the complete simplified dynamic model is chosen as $\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_a$.

Let $z_{x_0} = [\|e_{x_0}\|, \|\dot{e}_{x_0}\|]^T$, $z_{R_0} = [\|e_{R_0}\|, \|e_{\Omega_0}\|]^T$, $z_{q_i} = [\|e_{q_i}\|, \|e_{\omega_i}\|]^T \in \mathbb{R}^2$. The first part of the Lyapunov function \mathcal{V}_0 satisfies

$$\begin{aligned} z_{x_0}^T \underline{P}_{x_0} z_{x_0} + z_{R_0}^T \underline{P}_{R_0} z_{R_0} + \sum_{i=1}^n z_{q_i}^T \underline{P}_{q_i} z_{q_i} &\leq \mathcal{V}_0 \\ &\leq z_{x_0}^T \bar{P}_{x_0} z_{x_0} + z_{R_0}^T \bar{P}_{R_0} z_{R_0} + \sum_{i=1}^n z_{q_i}^T \bar{P}_{q_i} z_{q_i}, \end{aligned}$$

where the matrices $\underline{P}_{x_0}, \underline{P}_{R_0}, \underline{P}_{q_i}, \bar{P}_{x_0}, \bar{P}_{R_0}, \bar{P}_{q_i} \in \mathbb{R}^{2 \times 2}$ are given by

$$\begin{aligned} \underline{P}_{x_0} &= \frac{1}{2} \begin{bmatrix} k_{x_0} & -c_x \\ -c_x & 1 \end{bmatrix}, \quad \bar{P}_{x_0} = \frac{1}{2} \begin{bmatrix} k_{x_0} & c_x \\ c_x & 1 \end{bmatrix}, \\ \underline{P}_{R_0} &= \frac{1}{2} \begin{bmatrix} 2k_{R_0} & -c_R \bar{\lambda} \\ -c_R \bar{\lambda} & \bar{\lambda} \end{bmatrix}, \quad \bar{P}_{R_0} = \frac{1}{2} \begin{bmatrix} \frac{2k_{R_0}}{2 - \psi_{R_0}} & c_R \bar{\lambda} \\ c_R \bar{\lambda} & \bar{\lambda} \end{bmatrix}, \\ \underline{P}_{q_i} &= \frac{1}{2} \begin{bmatrix} 2k_q & -c_q \\ -c_q & 1 \end{bmatrix}, \quad \bar{P}_{q_i} = \frac{1}{2} \begin{bmatrix} \frac{2k_q}{2 - \psi_{q_i}} & c_q \\ c_q & 1 \end{bmatrix}, \end{aligned}$$

where $\bar{\lambda} = \lambda_m[J_0]$ and $\bar{\lambda} = \lambda_M[J_0]$. If the constants c_x, c_{R_0}, c_q are sufficiently small, all of the above matrices are positive-definite. As the second part of the Lyapunov function \mathcal{V}_a is already given as a quadratic form, it is straightforward to see that the complete Lyapunov function \mathcal{V} is positive-definite and decrescent.

The time-derivative of the Lyapunov function along the error dynamics (45), (47), and (49) is given by

$$\begin{aligned} \dot{\mathcal{V}} &= -(k_{\dot{x}_0} - c_x) \|\dot{e}_{x_0}\|^2 - c_x k_{x_0} \|e_{x_0}\|^2 - c_x k_{\dot{x}_0} e_{x_0} \cdot \dot{e}_{x_0} \\ &\quad + (c_x e_{x_0} + \dot{e}_{x_0}) \cdot Y_x - k_{\Omega_0} \|e_{\Omega_0}\|^2 + c_R \dot{e}_{R_0} \cdot J_0 e_{\Omega_0} \\ &\quad - c_R k_{R_0} \|e_{R_0}\|^2 + c_R e_{R_0} \cdot ((J_0 e_{\Omega_0} + d)^\wedge e_{\Omega_0} - k_{\Omega_0} e_{\Omega_0}) \\ &\quad + (e_{\Omega_0} + c_R e_{R_0}) \cdot Y_R \\ &\quad + \sum_{i=1}^n -(k_{\omega} - c_q) \|e_{\omega_i}\|^2 - c_q k_q \|e_{q_i}\|^2 - c_q k_{\omega} e_{q_i} \cdot e_{\omega_i} \\ &\quad + \frac{1}{m_0} (\dot{e}_{x_0} + c_x e_{x_0}) \cdot (\Phi_{x_0} \tilde{\theta}_{x_0} + \sum_{i=1}^n \Phi_{x_i} \tilde{\theta}_{x_i}) \\ &\quad + (e_{\Omega_0} + c_R e_{R_0}) \cdot (\Phi_{R_0} \tilde{\theta}_{R_0} + \sum_{i=1}^n \hat{\rho}_i R_0^T \Phi_{x_i}^{\parallel} \tilde{\theta}_{x_i}) \end{aligned}$$

$$\begin{aligned}
& - \left(\sum_{i=1}^n (e_{\omega_i} + c_q e_{q_i}) \cdot \frac{\hat{q}_i}{m_i l_i} \Phi_{x_i}^\perp \tilde{\theta}_{x_i} \right) - \frac{1}{h_{x_0}} \tilde{\theta}_{x_0} \cdot \dot{\tilde{\theta}}_{x_0} \\
& - \frac{1}{h_{R_0}} \tilde{\theta}_{R_0} \cdot \dot{\tilde{\theta}}_{R_0} - \sum_{i=1}^n \frac{1}{h_{x_i}} \tilde{\theta}_{x_i} \cdot \dot{\tilde{\theta}}_{x_i}. \quad (51)
\end{aligned}$$

In the above equation, the expressions at the last four lines depending on the estimate error can be rearranged by using the adaptive laws, (34)-(39) as

$$\begin{aligned}
& \tilde{\theta}_{x_0} \cdot (y_{x_0} - \Pr(\tilde{\theta}_{x_0}, y_{x_0})) + \tilde{\theta}_{R_0} \cdot (y_{R_0} - \Pr(\tilde{\theta}_{R_0}, y_{R_0})) \\
& + \sum_{i=1}^n \tilde{\theta}_{x_i} \cdot (y_{x_i} - \Pr(\tilde{\theta}_{x_i}, y_{x_i})).
\end{aligned}$$

From the definition of the projection map, the above expressions vanish for the first case of (33). For the second case,

$$\begin{aligned}
(\theta - \bar{\theta}) \cdot (y - \Pr(\bar{\theta}, y)) &= \frac{1}{\|\bar{\theta}\|^2} (\theta - \bar{\theta}) \cdot \bar{\theta} \bar{\theta}^T y \\
&= \frac{1}{\|\bar{\theta}\|^2} (\bar{\theta}^T \theta - \bar{\theta}^T \bar{\theta}) (\bar{\theta}^T y) \leq 0.
\end{aligned}$$

for each estimated parameter. The last inequality is due to $(\bar{\theta}^T \theta - \bar{\theta}^T \bar{\theta}) \leq 0$ and $(\bar{\theta}^T y) > 0$ obtained by (18) and (33).

An upper bound of the remaining expressions of \dot{V} at (51) can be obtained as follows. Since $\|e_{R_0}\| \leq 1$, $\|\dot{e}_{R_0}\| \leq \|e_{\Omega_0}\|$ and $\|d\| \leq B$,

$$\begin{aligned}
\dot{V} &\leq -(k_{\dot{x}_0} - c_x) \|\dot{e}_{x_0}\|^2 - c_x k_{x_0} \|e_{x_0}\|^2 - c_x k_{\dot{x}_0} e_{x_0} \cdot \dot{e}_{x_0} \\
&+ (c_x e_{x_0} + \dot{e}_{x_0}) \cdot Y_x - (k_{\Omega_0} - 2c_R \bar{\lambda}) \|e_{\Omega_0}\|^2 \\
&- c_R k_{R_0} \|e_{R_0}\|^2 + c_R (k_{\Omega_0} + B) \|e_{R_0}\| \|e_{\Omega_0}\| \\
&+ (e_{\Omega_0} + c_R e_{R_0}) \cdot Y_R \\
&+ \sum_{i=1}^n -(k_\omega - c_q) \|e_{\omega_i}\|^2 - c_q k_q \|e_{q_i}\|^2 - c_q k_\omega e_{q_i} \cdot e_{\omega_i}. \quad (52)
\end{aligned}$$

From (46), an upper bound of the fourth term of the right-hand side is given by

$$\begin{aligned}
& \|(c_x e_{x_0} + \dot{e}_{x_0}) \cdot Y_x\| \leq \\
& \sum_{i=1}^n \alpha_i \beta (c_x k_{x_0} \|e_{x_0}\|^2 + c_x k_{\dot{x}_0} \|e_{x_0}\| \|\dot{e}_{x_0}\| + k_{\dot{x}_0} \|\dot{e}_{x_0}\|^2) \\
& + \{c_x B \|e_x\| + (\beta k_{x_0} e_{x_{\max}} + B) \|\dot{e}_{x_0}\|\} \|e_{q_i}\| \\
& + \alpha_i \gamma (c_x \|e_{x_0}\| + \|\dot{e}_{x_0}\|) (k_{R_0} \|e_{R_0}\| + k_{\Omega_0} \|e_{\Omega_0}\|). \quad (53)
\end{aligned}$$

Similarly, using (48),

$$\begin{aligned}
& \|(c_R e_{R_0} + e_{\Omega_0}) \cdot Y_R\| \leq \\
& \sum_{i=1}^n \alpha_i \sigma_i (c_R k_{R_0} \|e_{R_0}\|^2 + c_R k_{\Omega_0} \|e_{R_0}\| \|e_{\Omega_0}\| + k_{\Omega_0} \|e_{\Omega_0}\|^2) \\
& + \{c_R B \|e_{R_0}\| + (\alpha_0 \sigma_i k_{R_0} + B) \|e_{\Omega_0}\|\} \|e_{q_i}\| \\
& + \alpha_i \delta_i (c_R \|e_{R_0}\| + \|e_{\Omega_0}\|) (k_{x_0} \|e_{x_0}\| + k_{\dot{x}_0} \|\dot{e}_{x_0}\|). \quad (54)
\end{aligned}$$

Substituting these into (52) and rearranging,

$$\dot{V} \leq \sum_{i=1}^n -z_i^T W_i z_i, \quad (55)$$

where $z_i = [\|z_{x_0}\|, \|z_{R_0}\|, \|z_{q_i}\|]^T \in \mathbb{R}^3$, and the matrix $W_i \in \mathbb{R}^{3 \times 3}$ is defined as

$$W_i = \begin{bmatrix} \lambda_m[W_{x_i}] & -\frac{1}{2}\|W_{xR_i}\| & -\frac{1}{2}\|W_{xq_i}\| \\ -\frac{1}{2}\|W_{xR_i}\| & \lambda_m[W_{R_i}] & -\frac{1}{2}\|W_{Rq_i}\| \\ -\frac{1}{2}\|W_{xq_i}\| & -\frac{1}{2}\|W_{Rq_i}\| & \lambda_m[W_{q_i}] \end{bmatrix}, \quad (56)$$

where the sub-matrices are given by

$$\begin{aligned}
W_{x_i} &= \frac{1}{n} \begin{bmatrix} c_x k_{x_0} (1 - n\alpha_i \beta) & -\frac{c_x k_{\dot{x}_0}}{2} (1 + n\alpha_i \beta) \\ -\frac{c_x k_{\dot{x}_0}}{2} (1 + n\alpha_i \beta) & k_{\dot{x}_0} (1 - n\alpha_i \beta) - c_x \end{bmatrix}, \\
W_{R_i} &= \frac{1}{n} \begin{bmatrix} c_R k_{R_0} (1 - n\alpha_i \sigma_i) & -\frac{c_R}{2} (k_{\Omega_0} + B + n\alpha_i \sigma_i) \\ -\frac{c_R}{2} (k_{\Omega_0} + B + n\alpha_i \sigma_i) & k_{\Omega_0} (1 - n\alpha_i \sigma_i) - 2c_R \bar{\lambda} \end{bmatrix}, \\
W_{q_i} &= \begin{bmatrix} c_q k_q & -\frac{c_q k_\omega}{2} \\ -\frac{c_q k_\omega}{2} & k_\omega - c_q \end{bmatrix}, \\
W_{xR_i} &= \alpha_i \begin{bmatrix} \gamma c_x k_{R_0} + \delta_i c_R k_{x_0} & \gamma c_x k_{\Omega_0} + \delta_i k_{x_0} \\ \gamma k_{R_0} + \delta_i c_R k_{\dot{x}_0} & \gamma k_{\Omega_0} + \delta_i k_{\dot{x}_0} \end{bmatrix}, \\
W_{xq_i} &= \begin{bmatrix} c_x B & 0 \\ \beta k_{x_0} e_{x_{\max}} + B & 0 \end{bmatrix}, \quad W_{xR_i} = \begin{bmatrix} c_R B & 0 \\ \alpha_0 \sigma_i k_{R_0} + B & 0 \end{bmatrix}.
\end{aligned}$$

If the constants c_x, c_R, c_q that are independent of the control input are sufficiently small, the matrices $W_{x_i}, W_{R_i}, W_{q_i}$ are positive-definite. Also, if the error in the direction of the link is sufficiently small relative to the desired trajectory, we can choose the controller gains such that the matrix W_i is positive-definite, which follows that the zero equilibrium of tracking errors is stable in the sense of Lyapunov, and all of the tracking error variables z_i and the estimation error variables are uniformly bounded, i.e., $e_{x_0}, \dot{e}_{x_0}, e_{R_0}, e_{\Omega_0}, e_{q_i}, e_{\omega_i}, \hat{\Delta}_{x_0}, \hat{\Delta}_{R_0}, \hat{\Delta}_{x_i} \in \mathcal{L}_\infty$. These also imply that $e_{x_0}, \dot{e}_{x_0}, e_{R_0}, e_{\Omega_0}, e_{q_i}, e_{\omega_i} \in \mathcal{L}_2$ from (55), and that $\dot{e}_{x_0}, \ddot{e}_{x_0}, \dot{e}_{R_0}, \dot{e}_{\Omega_0}, \dot{e}_{q_i}, \dot{e}_{\omega_i} \in \mathcal{L}_\infty$. According to Barbalat's lemma [12], all of the tracking error variables $e_{x_0}, \dot{e}_{x_0}, e_{R_0}, e_{\Omega_0}, e_{q_i}, e_{\omega_i}$ and their time-derivatives asymptotically converge to zero.

A. Proof of Proposition 2

We first show that the attitude of the i -th quadrotor becomes exactly equal to its desired value within a finite time, i.e., $R_i(t) = R_{i_c}(t)$ for any $t \geq T$ for some $T > 0$. This is achieved by finite-time stability theory [14]. This proof is composed of two parts: (i) $s_i(t) = 0$ for any $t > T_s$ for some $T_s < \infty$; (ii) when the state is confined to the surface defined by $s_i = 0$, we have $e_{R_i}(t) = e_{\Omega_i}(t) = 0$ for any $t > T_R$ for some $T_R < \infty$. From now on, we drop the subscript i for simplicity, as the subsequent development is identical for all quadrotors.

From [7], the error dynamics for e_Ω is given by

$$J \dot{e}_\Omega = -\Omega \times \Omega + M + \Delta_R + J(\hat{\Omega} R^T R_c \Omega_c - R^T R_c \dot{\Omega}_c). \quad (56)$$

Substituting (43),

$$\begin{aligned}
J \dot{e}_\Omega &= -k_s s - l_s S(r, s) + \Delta_R - B_\delta \frac{s}{\|s\|} \\
&- (k_R J + l_s r J \text{diag}_j [|e_{R_j}|^{r-1}]) E(R, R_c) e_\Omega. \quad (57)
\end{aligned}$$

Let a Lyapunov function be

$$\mathcal{W} = \frac{1}{2} s \cdot J s.$$

From (42) and (41), its time-derivative is given by

$$\dot{\mathcal{W}} = s \cdot \{J\dot{e}_\Omega + (k_R J + l_s r J \text{diag}_j[|e_{R_j}|^{r-1}])E(R, R_c)e_\Omega\}.$$

Substituting (57) and (43), and using (??), it reduces to

$$\begin{aligned} \dot{\mathcal{W}} &= s \cdot \{-k_s s - l_s S(r, s) + \Delta_R - \frac{s}{\|s\|} B_\delta\} \\ &\leq -k_s \|s\|^2 - l_s \sum_{j=1}^n |s_j|^{r+1} + B_\delta \|s\| - B_\delta \|s\| \\ &\leq -k_s \|s\|^2 - l_s \|s\|^{r+1}, \end{aligned}$$

where the last inequality is obtained from the fact that $\|x\|^\alpha \leq \sum_{i=1}^n |x_i|^\alpha$ for any $x = [x_1, \dots, x_n]^T$ and $0 < \alpha < 2$ [16, Lemma 2]. Therefore,

$$\dot{\mathcal{W}} \leq -\epsilon_1 \mathcal{W} - \epsilon_2 \mathcal{W}^{(r+1)/2},$$

where $\epsilon_1 = \frac{2k_s}{\lambda_M[J]}$ and $\epsilon_2 = l_s (\frac{2}{\lambda_M[J]})^{(r+1)/2}$. This implies that $s(t) = 0$ for any $t \geq T_s$, where the settling time T_s satisfies

$$T_s \leq \frac{2}{\epsilon_1(1-r)} \ln \frac{\epsilon_1 \mathcal{W}(0)^{(1-r)/2} + \epsilon_2}{\epsilon_2},$$

according to [15, Remark 2].

Next, consider the second part of the proof when $s = 0$. Let a configuration error function for the attitude of a quadrotor be

$$\Psi_R = \frac{1}{2} \text{tr}[I - R_c^T R].$$

Consider a domain give by $D_R = \{(R, \Omega) \in \text{SO}(3) \times \mathbb{R}^3 \mid \Psi_R < \psi_R < 2\}$. It has been shown that the following inequality is satisfied in the domain,

$$\frac{1}{2} \|e_R\|^2 \leq \Psi_R \leq \frac{1}{2 - \psi_R} \|e_R\|^2. \quad (58)$$

Therefore, it is positive-definite about $e_R = 0$. The time-derivative of Ψ_R is given by $\dot{\Psi}_R = e_R \cdot e_\Omega$. Therefore, when $s = 0$, we have

$$\begin{aligned} \dot{\Psi}_R &= -k_R \|e_R\|^2 - l_R \sum_{j=1}^n |e_{R_j}|^{r+1} \\ &\leq -k_R \|e_R\|^2 - l_R \|e_R\|^{r+1}, \end{aligned}$$

Substituting (58), we obtain

$$\dot{\Psi}_R \leq -\epsilon_3 \Psi_R - \epsilon_4 \Psi_R^{(r+1)/2},$$

where $\epsilon_3 = \frac{k_R}{2 - \psi_R}$ and $\epsilon_4 = \frac{l_R}{(2 - \psi_R)^{(r+1)/2}}$. This implies that $e_R(t) = e_\Omega(t) = 0$ for any $t \geq T_R$, where the settling time T_R satisfies

$$T_R \leq \frac{2}{\epsilon_3(1-r)} \ln \frac{\epsilon_3 \Psi_R(0)^{(1-r)/2} + \epsilon_4}{\epsilon_4}.$$

In summary, whenever $t \geq T^* \triangleq \max\{T_s, T_R\}$, it is guaranteed that $R_i(t) = R_{i_c}(t)$ for the i -th quadrotor. Next, we consider the *reduced system*, which corresponds to the dynamics of the payload and the rotational dynamics of the

links when $R_i(t) \equiv R_{i_c}(t)$. From (44) and (40), the control force of quadrotors when $R_i = R_{i_c}$ is given by

$$-f_i \cdot R_i e_3 = (u_i \cdot R_{c_i} e_3) R_{c_i} e_3 = (u_i \cdot \frac{u_i}{\|u_i\|}) - \frac{u_i}{\|u_i\|} = u_i.$$

Therefore, the reduced system is given by the controlled dynamics of the simplified model.

If the controller gains k_R, l_R, k_s, l_s are selected large such that T^* is sufficiently small, the solution stays inside of the domain D , where the stability results of Proposition 1 hold, during $0 \leq t < T^*$. After $t \geq T^*$, the controlled system corresponds to the controlled system of the simplified dynamic model, and from Proposition 1, the tracking errors asymptotically coverage to zero, and the estimation error are uniformly bounded.

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